Lecture 1: Introduction and Complex Number

1

Introduction to Quantum Computing

- History
- Particle-wave duality
- Superposition
- Quantum computer vs. Classic computer

2

Complex Number

- Motivation
- Definition

3

The Algebra Property

- Ordered pair representation
- Addition and multiplication
- Commutativity, associativity and distributive law
- Subtraction and division
- Modulus
- Conjugate

4

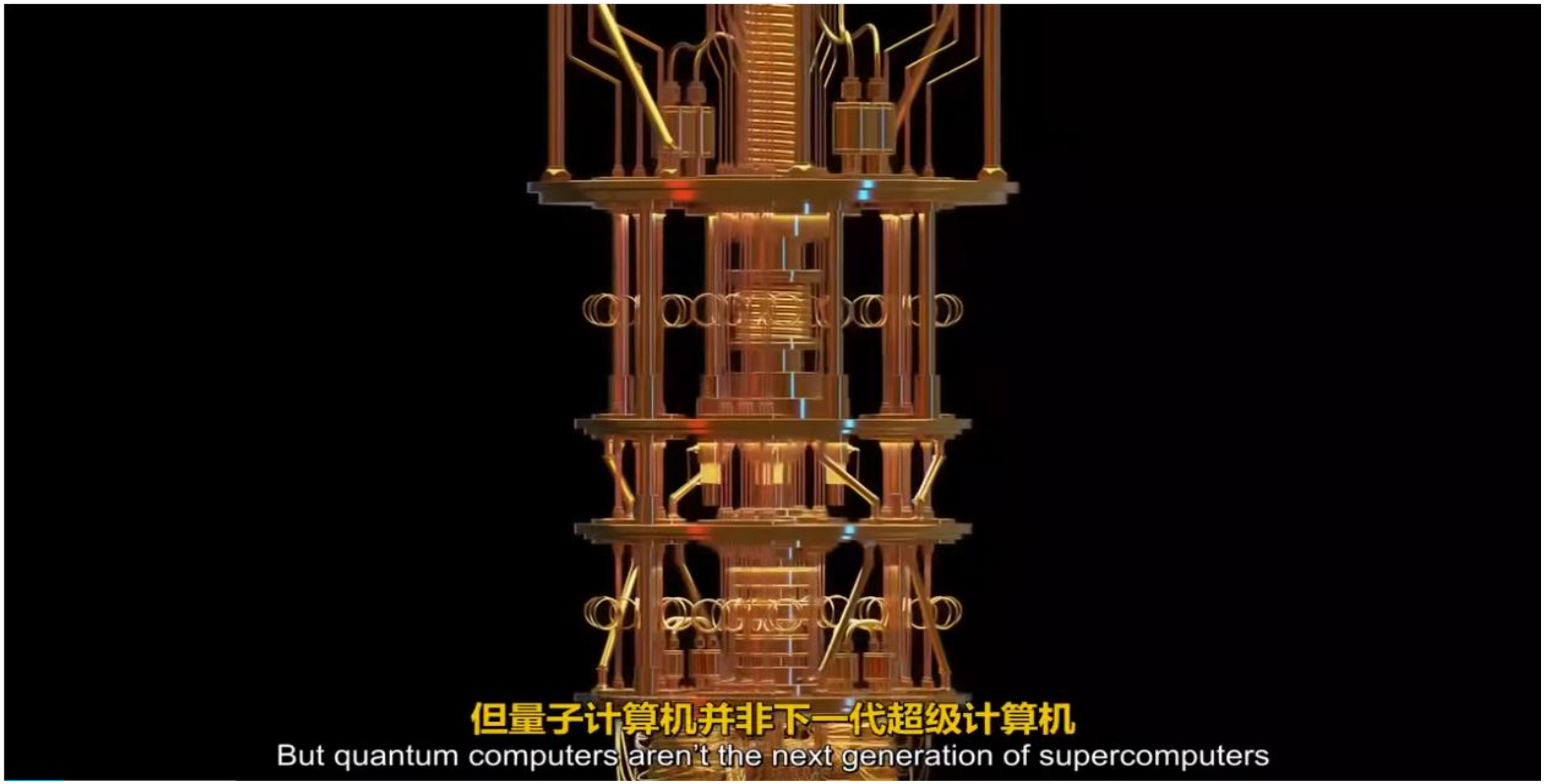
The Geometry Property

- Cartesian and polar representation
- Benefits of polar representation
- Cartesian-to-polar and polar-to-Cartesian representation

1. Introduction to QC

Source: Andrew C. Yao, The Advent of Quantum Computing, Micius Salon, No.8, 2018. https://www.bilibili.com/video/av33951287?from=search&seid=14096248465856158266

1. Introduction to QC



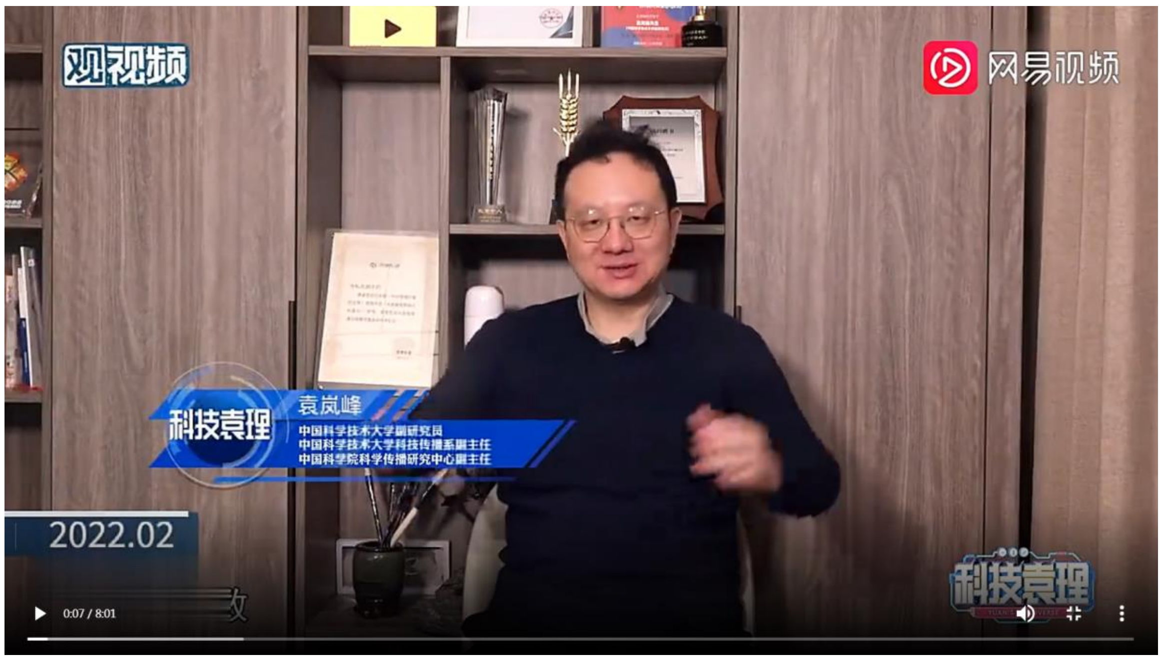
量子计算是宣传炒作吗？详细介绍量子计算以及费曼的开创性贡献
https://www.bilibili.com/video/BV1qg411G7hh/?spm_id_from=333.1007.top_right_bar_window_history.content.click&vd_source=322773747f9aa504da745054e83290e9

1. Introduction to QC

潘建伟：从爱因斯坦的好奇心到量子信息革命 | 2020新年科学演讲全程
https://www.bilibili.com/video/BV1s7411v7ZR?from=search&seid=7726410632245204046&spm_id_from=333.337.0.0

2. Complex Number

- 量子力学必须是复数的



中国科学家实验确认，量子力学必须是复数的
https://www.163.com/v/video/VZVJ8JI1Q.html

2. Complex Number

- Motivation
  - Algebraic equation

$$x^2+1=0$$

  - No solutions in the following number sets
    - positive numbers, $\mathbb{P}=\{1,2,\cdots\}$
    - natural numbers, $\mathbb{N}=\{0,1,2,\cdots\}$
    - integers (or whole numbers), $\mathbb{Z}=\{-2,-1,0,1,2,\cdots\}$
    - rational numbers, $\mathbb{Q}=\{\frac{m}{n}|m\in\mathbb{Z},n\in\mathbb{P}\}$
    - real numbers, $\mathbb{R}=\mathbb{Q}\cup\{\cdots,\sqrt{2},\cdots,e,\cdots,\pi,\cdots,\frac{e}{\pi},\cdots\}$

（感谢弘毅学堂 2018级耿一洋同学指正此页 $\mathbb{Q}$ 与 $\mathbb{Z}$ 的符号错误）

2. Complex Number

- Definitions
  - Imaginary number i such that

$$i^2 = -1 \text{ or } i = \sqrt{-1}$$

  - Complex number $c \in \mathbb{C}$ such that

$$c = a + b \times i = a + bi$$

    - $a \in \mathbb{R}$ is the real part of c
    - $b \in \mathbb{R}$ is the imaginary part of c
    - $\mathbb{C}$ denotes the complex number set

2. Complex Number

- Example

Example 1.1.2 Let $c_1 = 3 - i$ and $c_2 = 1 + 4i$. We want to compute $c_1 + c_2$ and $c_1 \times c_2$.

$$c_1 + c_2 = 3 - i + 1 + 4i = (3 + 1) + (-1 + 4)i = 4 + 3i. \tag{1.6}$$

Multiplying is not as easy. We must remember to multiply each term of the first complex number with each term of the second complex number. Also, remember that $i^2 = -1$.

$$\begin{aligned} c_1 \times c_2 &= (3 - i) \times (1 + 4i) = (3 \times 1) + (3 \times 4i) + (-i \times 1) + (-i \times 4i) \\ &= (3 + 4) + (-1 + 12)i = 7 + 11i. \end{aligned} \tag{1.7}$$

□

2. Complex Number

- Proposition (命题)

Proposition 1.1.1 (Fundamental Theorem of Algebra). Every polynomial equation of one variable with complex coefficients has a complex solution.

3. The Algebra Property

- Ordered-Pair representation*

$$c = a + bi \mapsto (a, b)$$

- Examples
  - Ordinary real number $a = a + 0 \cdot i \mapsto (a, 0)$
  - Imaginary number $i = 0 + 1 \cdot i \mapsto (0, 1)$

* It is not a vectorization. See its multiplication operation.

3. The Algebra Property

- Addition
  - It add pairs componentwise

$$c_1 + c_2 = (a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$$

- Multiplication

$$c_1 \times c_2 = (a_1, b_1) \times (a_2, b_2) = (a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1)$$

- Note

$$c = a + bi = (a, b) = (a, 0) + (0, b)$$

$$= (a, 0) + (b, 0) \times (0, 1)$$

3. The Algebra Property

- Example

Example 1.2.1 Let $c_1 = (3, -2)$ and $c_2 = (1, 2)$. Let us multiply them using the aforementioned rule:

$$\begin{aligned} c_1 \times c_2 &= (3 \times 1 - (-2) \times 2, -2 \times 1 + 2 \times 3) \\ &= (3 + 4, -2 + 6) = (7, 4) = 7 + 4i. \end{aligned} \tag{1.15}$$

□

- Additive identity （加法单位元）

$$\forall c \in \mathbb{C},\ c + (0, 0) = c$$

- Multiplicative identity （乘法单位元）

$$\forall c \in \mathbb{C},\ c \times (1, 0) = (1, 0) \times c = c$$

3. The Algebra Property

- Commutativity
  - Both operations are commutative

$$c_1 + c_2 = c_2 + c_1 \quad \text{and} \quad c_1 \times c_2 = c_2 \times c_1$$

- Associativity
  - Both operations are associative

$$(c_1 + c_2) + c_3 = c_1 + (c_2 + c_3) \ \text{and} \ (c_1 \times c_2) \times c_3 = c_1 \times (c_2 \times c_3)$$

- Distributive property (try to prove)
  - multiplication distributes over addition

$$c_1 \times (c_2 + c_3) = c_1 \times c_2 + c_1 \times c_3$$

3. The Algebra Property

■ Proof for distributive law

Let us verify this property: first we write the complex numbers as pairs $c_1 = (a_1, b_1)$, $c_2 = (a_2, b_2)$, and $c_3 = (a_3, b_3)$. Now, let us expand the left side

$$\begin{aligned} c_1 \times (c_2 + c_3) &= (a_1, b_1) \times ((a_2, b_2) + (a_3, b_3)) \\ &= (a_1, b_1) \times (a_2 + a_3, b_2 + b_3) \\ &= (a_1 \times (a_2 + a_3) - b_1 \times (b_2 + b_3), \\ &\quad a_1 \times (b_2 + b_3) + b_1 \times (a_2 + a_3)) \\ &= (a_1 \times a_2 + a_1 \times a_3 - b_1 \times b_2 - b_1 \times b_3, \\ &\quad a_1 \times b_2 + a_1 \times b_3 + b_1 \times a_2 + b_1 \times a_3). \end{aligned} \tag{1.21}$$

Turning to the right side of Equation (1.20) one piece at a time gives

$$c_1 \times c_2 = (a_1 \times a_2 - b_1 \times b_2, a_1 \times b_2 + a_2 \times b_1) \tag{1.22}$$

$$c_1 \times c_3 = (a_1 \times a_3 - b_1 \times b_3, a_1 \times b_3 + a_3 \times b_1); \tag{1.23}$$

summing them up we obtain

$$\begin{aligned} c_1 \times c_2 + c_1 \times c_3 = (&a_1 \times a_2 - b_1 \times b_2 + a_1 \times a_3 - b_1 \times b_3, \\ &a_1 \times b_2 + a_2 \times b_1 + a_1 \times b_3 + a_3 \times b_1), \end{aligned} \tag{1.24}$$

which is precisely what we got in Equation (1.21).

3. The Algebra Property

- Subtraction

$$c_1 - c_2 = (a_1, b_1) - (a_2, b_2) = (a_1 - a_2, b_1 - b_2)$$

- Division (try to prove)

$$\frac{c_1}{c_2} = \frac{(a_1, b_1)}{(a_2, b_2)} = \left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2}, \frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2} \right)$$

（感谢弘毅学堂 2018级彭可心同学指正此页除法公式错误）

3. The Algebra Property

- Partial proof for division equation

As for division, we have to work a little: If

$$(x, y) = \frac{(a_1, b_1)}{(a_2, b_2)}, \tag{1.26}$$

then by definition of division as the inverse of multiplication

$$(a_1, b_1) = (x, y) \times (a_2, b_2) \tag{1.27}$$

or

$$(a_1, b_1) = (a_2 x - b_2 y, a_2 y + b_2 x). \tag{1.28}$$

So we end up with

$$(1) \qquad a_1 = a_2 x - b_2 y, \tag{1.29}$$

$$(2) \qquad b_1 = a_2 y + b_2 x. \tag{1.30}$$

3. The Algebra Property

- Absolute value of a real number

$$|a| = +\sqrt{a^2}$$

- Modulus of a complex number

$$|c| = |a+bi| = +\sqrt{a^2+b^2}$$

$$\Leftrightarrow \ |c|^2 = a^2+b^2$$

  - Property 1: $\forall c_1, c_2 \in \mathbb{C},\ |c_1|\,|c_2| = |c_1 c_2|$
  - Property 2: $\forall c_1, c_2 \in \mathbb{C},\ |c_1 + c_2| \leqslant |c_1| + |c_2|$

（感谢弘毅学堂 2018级宋文卓同学指正此页Property 2的公式错误）

3. The Algebra Property

- Conjugation
  - Change the sign of the imaginary part

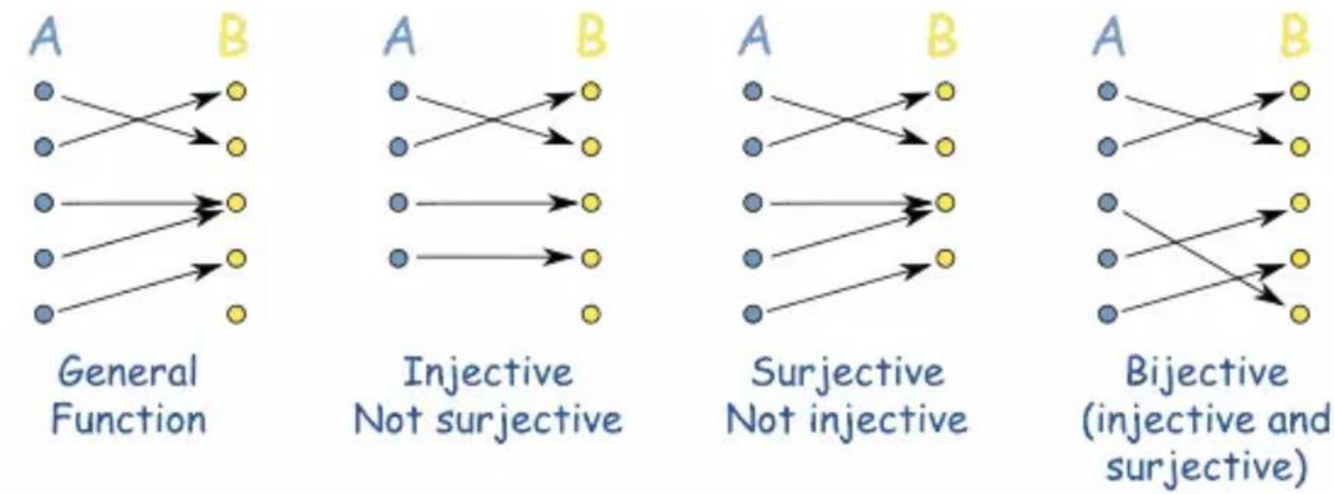
$$\text{original:}\ \ c = a + bi$$

$$\text{conjugate:}\ \ \overline{c} = a - bi$$

    - Conjugate respects addition $\overline{c_1 + c_2} = \overline{c_1} + \overline{c_2}$
    - Conjugate respects multiplication $\overline{c_1 \times c_2} = \overline{c_1} \times \overline{c_2}$
  - Conjugation $c \longmapsto \overline{c}$ is bijective

Supplementary material

- Injective（单射）, surjective（满射） and bijective（双射）

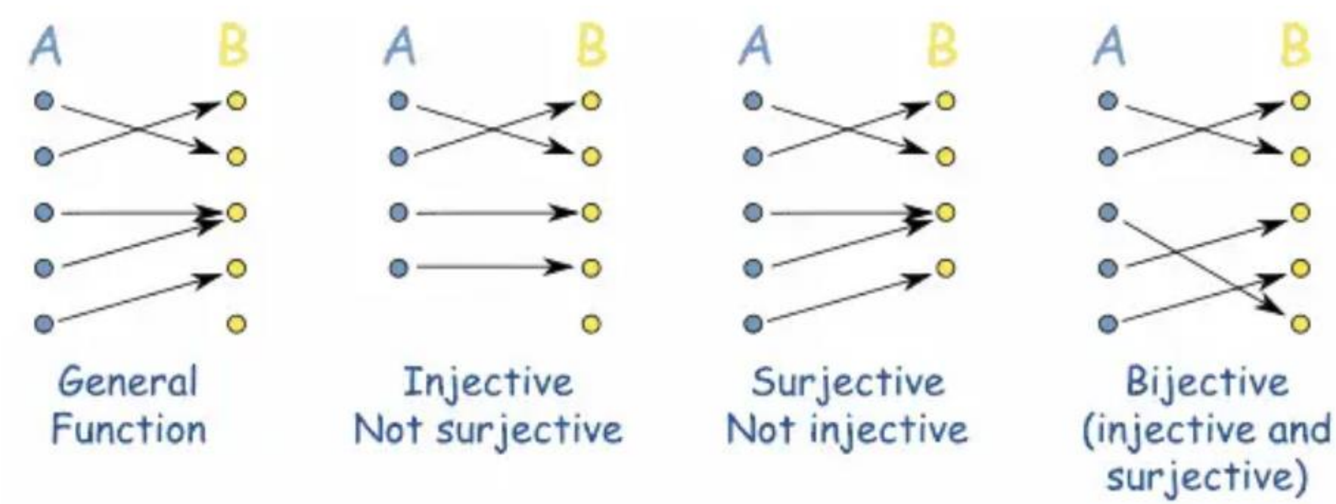


  - Injective means every “A” has its own unique matching member in “B”
  - Injective is strictly 1:1 (1:N and N:1 are NOT OK)

Image source: https://www.jianshu.com/p/09e6df559970

Supplementary material

- Injective（单射）, surjective（满射） and bijective（双射）



- Surjective means every “B” has at least one matching “A”
- Bijective means both injective and surjective

Image source: https://www.jianshu.com/p/09e6df559970

4. The Geometry Property

- Complex plane and parallelogram rule
  - The Cartesian representation

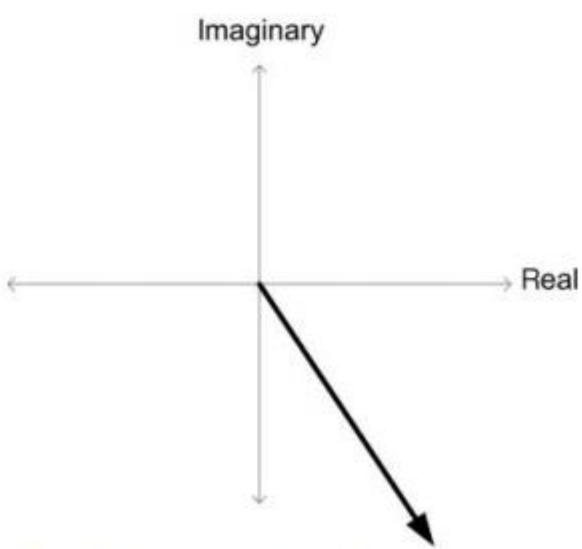


Figure 1.1. Complex plane.

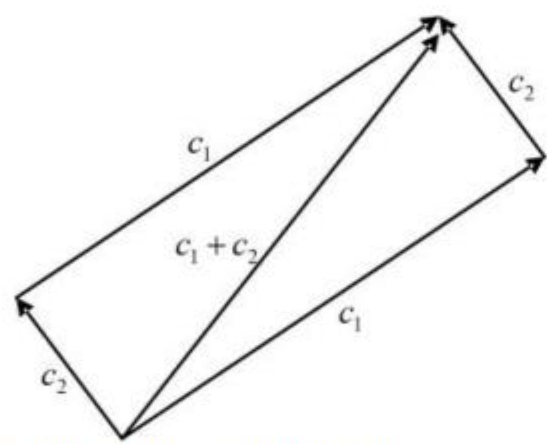


Figure 1.3. Parallelogram rule.

Cartesian 为什么翻译成笛卡尔?
https://www.zhihu.com/question/23903885/answer/26029956

4. The Geometry Property

- From Cartesian-to-Polar representation

$$c = a + bi \longmapsto (a, b) \longmapsto (\rho, \theta)$$

$$\text{where} \quad \rho = \sqrt{(a^2 + b^2)}$$

$$\text{and} \quad \theta = \text{atan}2(b, a) \in (-\pi, \pi]$$

- Some alias
  - ρ : length, magnitude
  - θ : time, phase

（感谢弘毅学堂 2018 级曹露瑶同学指正此页$\tan^{-1}$公式中的a与b位置错误）

（感谢弘毅学堂 2020 级王蕴飞同学指正此页$\tan^{-1}$公式值域范围错误）

补充说明

■ 关于atan2函数的说明

- atan2是一个函数，在C语言里返回的是指方位角，C 语言中atan2的函数原型为 double atan2(double y, double x)，返回以弧度表示的 y/x 的反正切。y 和 x 的值的符号决定了正确的象限。也可以理解为计算复数 x+yi 的辐角

$$\operatorname{atan2}(y,x)=\begin{cases}\arctan\left(\frac{y}{x}\right) & x>0\\ \arctan\left(\frac{y}{x}\right)+\pi & y\geq 0, x<0\\ \arctan\left(\frac{y}{x}\right)-\pi & y<0, x<0\\ +\frac{\pi}{2} & y>0, x=0\\ -\frac{\pi}{2} & y<0, x=0\\ \text{undefined} & y=0, x=0\end{cases}$$

参考资料：atan2，https://baike.baidu.com/item/atan2/10931300?fr=aladdin

4. The Geometry Property

- Why Polar representation
  - For fast multiplication and division

$$\text{multiplication: } c_1 \times c_2 = (\rho_1, \theta_1) \times (\rho_2, \theta_2) = (\rho_1 \rho_2,\ \theta_1 + \theta_2)$$

$$\text{division: } \frac{c_1}{c_2} = \left(\frac{\rho_1}{\rho_2},\ \theta_1 - \theta_2\right)$$

  - For fast power and root calculation

$$n^{\text{th}} \text{ power: } c^n = (\rho^n, n\theta)$$

$$n^{\text{th}} \text{ root: } c^{\frac{1}{n}} = \left(\rho^{\frac{1}{n}}, \frac{1}{n}(\theta + k2\pi)\right), \quad k = 0, 1, \cdots, n-1$$

4. The Geometry Property

- Benefits of Polar representation
  - Multiplication : $c_1 \times c_2 = (\rho_1, \theta_1) \times (\rho_2, \theta_2) = (\rho_1\rho_2, \theta_1 + \theta_2)$
- Example

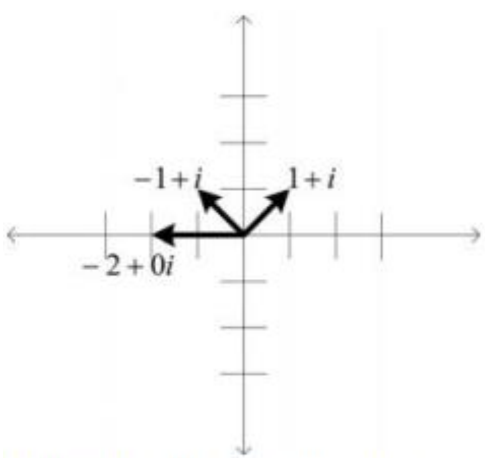


Figure 1.7. Two complex numbers and their product.

Example 1.3.4 Let $c_1 = 1 + i$ and $c_2 = -1 + i$. Their product, according to the algebraic rule, is

$$c_1c_2 = (1 + i)(-1 + i) = -2 + 0i = -2. \quad (1.61)$$

Now, let us take their polar representation

$$c_1 = \left(\sqrt{2}, \frac{\pi}{4}\right), \qquad c_2 = \left(\sqrt{2}, \frac{3\pi}{4}\right). \quad (1.62)$$

(Carry out the calculations!) Therefore, their product using the rule described earlier is

$$c_1c_2 = \left(\sqrt{2} \times \sqrt{2}, \frac{\pi}{4} + \frac{3\pi}{4}\right) = (2, \pi). \quad (1.63)$$

If we revert to its Cartesian coordinates, we get

$$(2 \times \cos(\pi), 2 \times \sin(\pi)) = (-2, 0), \quad (1.64)$$

which is precisely the answer we arrived at with the algebraic calculation in Equation (1.61).

4. The Geometry Property

- Benefits of Polar representation
  - Division : $\frac{c_1}{c_2} = \left(\frac{\rho_1}{\rho_2}, \theta_1 - \theta_2\right)$
- Example

Example 1.3.5 Let $c_1 = -1 + 3i$ and $c_2 = -1 - 4i$. Let us calculate their polar coordinates first:

$$c_1 = \left(\sqrt{(-1)^2 + 3^2}, \tan^{-1}\left(\frac{3}{-1}\right)\right) = (\sqrt{10}, \tan^{-1}(-3)) = (3.1623, 1.8925), \tag{1.70}$$

$$c_2 = \left(\sqrt{(-1)^2 + (-4)^2}, \tan^{-1}\left(\frac{-4}{-1}\right)\right) = (\sqrt{17}, \tan^{-1}(4)) = (4.1231, -1.8158), \tag{1.71}$$

therefore, in polar coordinates the quotient is

$$\frac{c_1}{c_2} = \left(\frac{3.1623}{4.1231}, 1.8925 - (-1.8158)\right) = (0.7670, 3.7083). \tag{1.72}$$

□

4. The Geometry Property

- Benefits of Polar representation
  - n-th power : $c^n = (\rho^n, n\theta)$
- Example

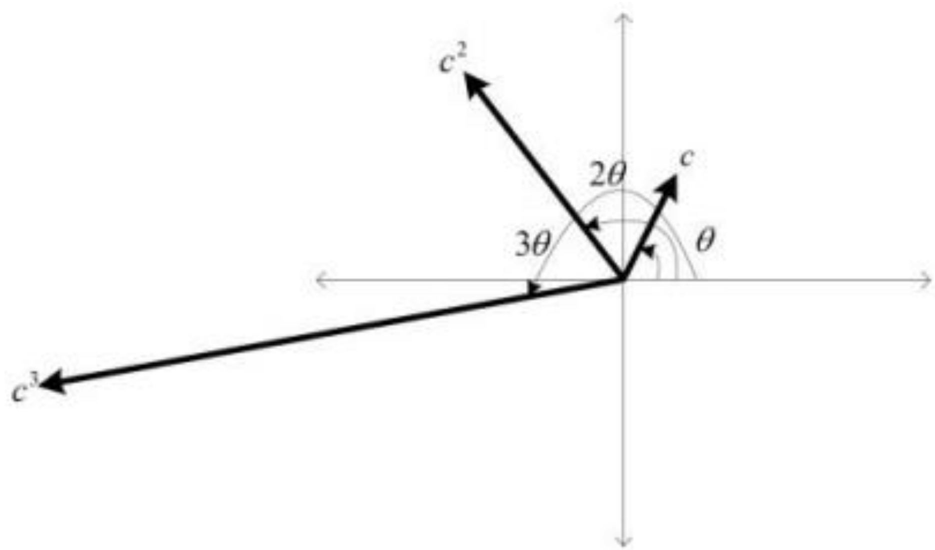


Figure 1.9. A complex number and its square and cube.

4. The Geometry Property

- Benefits of Polar representation
  - n-th root : $c^{\frac{1}{n}} = \left(\rho^{\frac{1}{n}}, \frac{1}{n}(\theta + k2\pi)\right), \quad k = 0, 1, \cdots, n-1$
- Examples

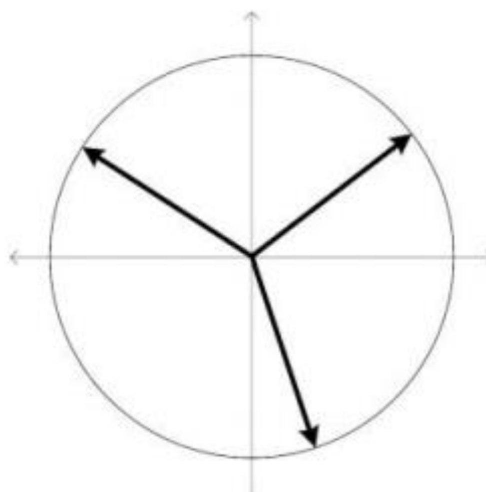


Figure 1.10. The three cube roots of unity.

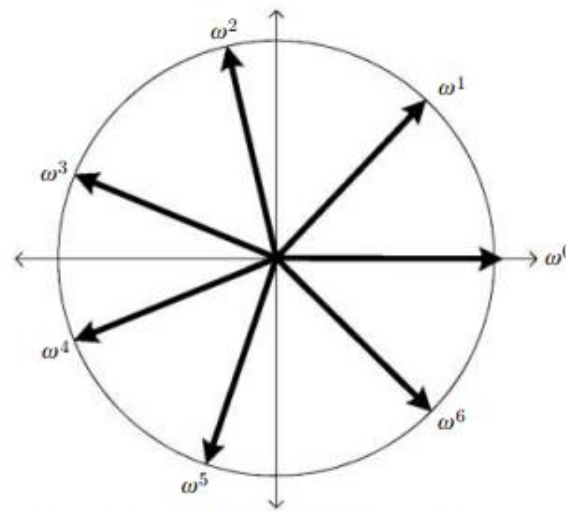


Figure 1.11. The seventh root of unity and its powers.

（感谢人工智能专业2022级张烨同学指出本页存在多余字符的错误）

4. The Geometry Property

- From Polar-to-Cartesian representation

$$c = \rho(\cos(\theta) + i\ \sin(\theta))$$

- Euler Equation

$$e^{i\theta} = \cos(\theta) + i\ \sin(\theta)$$

- New Cartesian Representation

$$c = \rho e^{i\theta}$$

Conclusion

1. Introduction to Quantum Computing
   - Superposition
   - Quantum Computer vs. Classic Computer
2. Complex Number
3. The Algebra Property
   - Ordered pair representation
   - Modulus
   - conjugate
4. The Geometry Property
   - Benefits of polar representation